Section 4

1. Let's assume we want to design a biased coin for some surreptitious purpose. Fortunately, we have a machine that will create coins that land on heads with probability p and tails with probability $1-p$. How should we set p so that after flipping the coin $h+t$ times the outcome with maximum probability is getting h heads and t tails? As a reminder, the probability is

$$\mathbf{Pr}[h \text{ heads and } t \text{ tails}] = \binom{h+t}{h} p^h (1-p)^t$$

To determine the maximum, you will need use some calculus, *i.e.* take the derivative with respect to p, set it to 0, and solve for p. Assume that $h \geq 1$ and $t \geq 1$ are fixed constants (in other words $\frac{d(p^h)}{dp} = hp^{h-1}$, etc). *Hint*: the answer you get for p should be fairly intuitive.

2. For any two (not necessarily independent) random variables X and Y, prove that

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y].$$

As a reminder, the *expected value* is defined as

$$\mathbf{E}[X] = \sum_{i=1}^{m} x_i \cdot \mathbf{Pr}[X = x_i].$$

where in this case X only takes values in $\{x_1, \ldots, x_m\} \subseteq \mathbb{R}$.

3. Prove *Markov's inequality:* for any nonnegative random variable X, and for any $t \geq 0$,

$$\mathbf{Pr}[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.$$

Then, show the equivalence to the sometimes more useful form

$$\mathbf{Pr}[X \geq t \cdot \mathbf{E}[X]\,] \leq \frac{1}{t}.$$

Finally, invent a random variable and a distribution such that,

$$\mathbf{Pr}[X \geq 10 \cdot \mathbf{E}[X]\,] = \frac{1}{10}.$$

4. Notice that for any function g and random variable X, the function $g(X)$ is also a random variable.

   (a) Write down the formula for $\mathbf{E}[g(X)]$.

   (b) Verify the natural equality:

$$\mathbf{E}[g(X)] = \sum_{i=1}^{m} g(x_i)\mathbf{Pr}[X = x_i].$$

   (c) Come up with an example where you might care about $g(X)$ instead of just X.